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IN THE THEORY OF THE TURBULENT DIFFUSION OF A HEAVY  
AEROSOL IN THE BOUNDARY LAYER OF THE ATMOSPHERE

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ON THE HYPOTHESIS OF A QUASI-STATIONARY SURFACE LAYER  
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AEROSOL IN THE BOUNDARY LAYER OF THE ATMOSPHERE

[ Following is the translation of "O gipoteze kvazi-statsionarnogo prizemnogo sloya v teorii turbulentnoy diffusii tyazhelogo aerolya v pograničnom sloye atmosfery" (English version above) by I.L. Karol' in Izv. Akad. Nauk SSSR, ser. geofiz. (Trans. AS USSR, geophys. ser.), No. 5, 1960, pages 758-761.]

1. In papers [1] and [2] on the semi-empirical theory of turbulence, we examined the non-stationary process of vertical turbulent diffusion of a heavy homogeneous aerosol from an instantaneous point source, placed high above the surface layer of the atmosphere. Since the exact solution of this problem is tedious and of little use for analysis and numerical calculation, a simple approximate solution was described in [1], in which the variation of the coefficient of vertical turbulent diffusion in the surface layer was taken into account in the boundary condition for the problem at ground level.

The exact solution of the problem can be simplified in another way. In papers [3], [4] and [5] on the theory of temperature distribution and the thermal transformation of air masses in the boundary layer of the atmosphere, it was shown

that the temperature change in the surface layer can be considered as quasi-stationary from some time sufficiently long after the beginning of the process, i.e. we can ignore the partial derivatives with respect to time in the corresponding differential equations.

In this paper we discuss the possibility of applying the hypothesis of a quasi-stationary surface layer to the problem of the turbulent diffusion of a heavy aerosol, and the connection between the approximate solution obtained in this way and the approximate solution found in [1].

In [1] and [2], the dimensionless concentration  $q(\tau, \zeta)$  of the aerosol is defined as that solution of the equation

$$\frac{\partial q}{\partial \tau} - \nu \frac{\partial q}{\partial \zeta} = \frac{\partial}{\partial \zeta} K(\zeta) \frac{\partial q}{\partial \zeta}; \quad (1.1)$$

$$K(\zeta) = \begin{cases} \zeta; & \zeta \leq 1 \\ 1; & \zeta > 1, \end{cases}$$

which satisfies the initial and boundary conditions

$$q(0, \zeta) = \delta(\zeta - \chi) \quad (1.2)$$

$$q \rightarrow 0, \quad \zeta \rightarrow +\infty; \quad [K(\zeta) \partial q / \partial \zeta - \theta q]_{\zeta=\zeta_0} = 0, \quad \theta = \beta - \nu. \quad (1.3)$$

We shall use the notation of [1] and [2]:  $\tau, \zeta, \zeta_0, \chi, \nu = \omega, \beta$  are, respectively, dimensionless time, the vertical coordinate, the height of the 'unevenness layer', the height of the source, the velocity of the gravitational subsidence of the aerosol, and a parameter characterising the reciprocal action of the aerosol and the surface of the earth ( $\beta = 0$  - reflection,  $\beta = \infty$  - absorption of the aerosol by the earth). The height of the surface layer is taken to be equal to unity and to

be greater than  $X > 1^*$ .

Taking the diffusion in the surface layer to be quasi-stationary, we replace (1.1) by the equation

$$\frac{\partial}{\partial \zeta} K(\zeta) \frac{\partial q_1^{(1)}}{\partial \zeta} + \nu \frac{\partial q_1^{(1)}}{\partial \zeta} = 0, \quad \zeta_0 < \zeta < 1; \quad \frac{\partial q_1^{(2)}}{\partial \tau} - \nu \frac{\partial q_1^{(2)}}{\partial \zeta} = \frac{\partial^2 q_1^{(2)}}{\partial \zeta^2}, \quad \zeta > 1. \quad (1.1a)$$

In this case the concentration  $q_1(\tau, \zeta)$  of the aerosol is found by solving the problem (1.1a) - (1.3) in exactly the same way as the problem (1.1) - (1.3) was solved in [1] and [2], in particular by using a one-sided Laplace transformation for the time  $\tau$  and by subsequent asymptotic representation of the Mellin rotation integral for large values of  $\tau$  and with  $\chi_1 = \chi - 1$ . In this way we obtain the following asymptotic representation for  $q_1^{(2)}(\tau, \zeta)$  for  $1 < \zeta < \chi$ :

$$q_1^{(2)}(\tau, \zeta) = \frac{\chi_1 e^{-(\chi_1 - \nu\tau)^2/4\tau}}{2\sqrt{\pi\tau^3}(\lambda_1 - \lambda_2)} [e^{\lambda_1(\zeta-1)} - e^{\lambda_2(\zeta-1)} \Psi_{\nu 1}(p_0, \zeta_0, \theta)] \left[ 1 + O\left(\frac{\chi_1^2}{\tau^3}\right) \right], \quad (1.4)$$

where

$$\text{the } \lambda_{1,2} = -\nu/2 \pm \sqrt{\nu^2/4 + p_0}; \quad p_0 = (\chi_1^2 \tau^{-2} - \nu^2)/4;$$

$$\Psi_{\nu 1}(p_0, \zeta_0, \theta) = \frac{(\lambda_1 + \nu)\theta \zeta_0^\nu - \lambda_1 \beta}{(\lambda_2 + \nu)\theta \zeta_0^\nu - \lambda_2 \beta}, \quad \nu > 0; \quad \Psi_{01}(p_0, \zeta_0, \beta) = \frac{\lambda_1(1 - \beta \ln \zeta_0) - \beta}{\lambda_2(1 - \beta \ln \zeta_0) - \beta}, \quad \nu = 0. \quad (1.5)$$

The solution  $q^2(\tau, \zeta)$  of the problem (1.1) - (1.3) in the layer  $1 < \zeta < \chi$  also has the asymptotic representation (1.4), but the function  $\Psi_\nu(p_0, \zeta_0, \theta)$  is then a complicated expression (see [2]) involving the modified Bessel functions  $I_\nu(2\sqrt{p_0})$ ,  $K_\nu(2\sqrt{p_0})$ ,  $I_\nu(2\sqrt{p_0 \zeta_0})$ ,  $K_\nu(2\sqrt{p_0 \zeta_0})$ .

\* In [2] we also discussed the case when there is a steady change in the coefficient of vertical turbulent diffusion in the surface layer,  $K(\zeta) = \zeta^\epsilon$ ,  $\zeta < 1$ ,  $0 < \epsilon < 2$ , for a 'weightless compound' ( $\omega = 0$ ).

Let us consider the case  $\nu=0$ . If we take  $\chi_1$ , to be bounded and let  $\tau \rightarrow \infty$ , then  $p_0 = \chi_1^2/4\tau^2 \rightarrow 0$ ,  $\lambda_{1,2} = \pm \sqrt{p_0} \rightarrow 0$ . Then, by using the expression for Bessel functions for small arguments, it is easy to see that in this case  $\Psi_0(p_0, \zeta_0, \beta)$  becomes  $\Psi_{01}(p_0, \zeta_0, \beta)$

i.e. the principal term in the asymptotic representation of  $q^2(\tau, \zeta)$ , the solution of the 'exact' problem (1.1) - (1.3), is, for large values of  $\tau$ , equal to the principal term of  $q_1^{(2)}(\tau, \zeta)$ , the solution of (1.1a) - (1.3), the problem with a quasi-stationary surface layer\*.

We shall now take  $\nu > 0$ , and, for simplicity, we put  $\zeta_0 = 0$ \*\*. Then the expression will take the form (see [2]):

$$\Psi_\nu(p_0, 0, 0) = \frac{\sqrt{p_0} I_{\nu+1/2}(2\sqrt{p_0}) - \lambda_1 I_\nu(2\sqrt{p_0})}{\sqrt{p_0} I_{\nu+1/2}(2\sqrt{p_0}) - \lambda_2 I_\nu(2\sqrt{p_0})} \quad (1.6)$$

As  $\tau \rightarrow \infty$  and for bounded  $\chi_1$ , since  $p_0 \rightarrow -\nu^2/4 \neq 0$ ,  $\lambda_{1,2} \rightarrow -\nu/2$  in this case (1.6) does not become equal to  $\Psi_{\nu 1}(p_0, 0, 0) = \lambda_1/\lambda_2$  for  $q_1^{(2)}(\tau, \zeta)$ , the solution of (1.1a) - (1.3), although as  $\tau \rightarrow \infty$  the limits of these expressions are equal,  $\lim_{\tau \rightarrow \infty} \Psi_\nu(p_0, 0, 0) = \lim_{\tau \rightarrow \infty} \Psi_{\nu 1}(p_0, 0, 0) = 1$ . It is easy to see that this also happens when  $\zeta_0 > 0$  and for any values of the parameter  $\beta$ .

\* There is a similar result for a 'weightless' compound in the case  $K(\zeta) = \zeta^2$ ,  $0 < \zeta < 2$  in the surface layer.

\*\*The possibility of ignoring the height of the 'unevenness layer' is examined in [1] and [2], where it is established that if this is done the solution does not depend on the parameter  $\beta$ .

Thus, when we are concerned with the diffusion of a heavy compound ( $\nu > 0$ ) the principal term in the asymptotic representation of the solution  $q^{(2)}(\tau, \zeta)$  of the 'exact' problem (1.1) - (1.3) does not become equal to that of  $q_1^{(2)}(\tau, \zeta)$ , the solution of the problem with a quasi-stationary surface layer, for large values of  $\tau$ , although, as  $\tau \rightarrow \infty$ , these solutions have the same limit: equal either to zero or to the stationary distribution of concentration in height when the aerosol is reflected by the earth's surface.

Thus, the simplifying assumption for the problem (1.1) - (1.3) that the change of concentration of the aerosol in the surface layer is quasi-stationary can be made for large values of  $\tau$  only when  $\nu = 0$ , i.e. only for a 'weightless' aerosol\*.

2. To calculate the time taken to set up quasi-stationary state in the surface layer when  $\nu = 0$  we define the relative error  $E$  of substituting  $q^{(2)}(\tau, \zeta)$  by  $q_1^{(2)}(\tau, \zeta)$ , putting

$$E(\tau, \zeta) = 1 - q^{(2)}(\tau, \zeta) / q_1^{(2)}(\tau, \zeta) \quad (2.1)$$

and restricting ourselves to the case  $\zeta_0 = 0$ . Putting the principal values of  $q^{(2)}$  and  $q_1^{(2)}$  given in (1.4) in  $\Psi_0(p_0, 0, \beta)$  defined by (1.6) with  $\nu = 0$  and  $\Psi_{01}(p_0, 0, \beta) = -1$ , after simplification we have, on the boundary of the surface layer ( $\zeta = 1$ ):

$$E(\tau, 1) = I_0(x) / [I_1(x) + I_0(x)]; \quad x = \chi_1 / \tau.$$

The requirement  $|E| < \varepsilon$  gives an inequality for  $x$ :

$$I_0(x) / I_1(x) > \varepsilon^{-1} - 1. \quad (2.2)$$

\* [3] - [5] concern particularly the turbulent diffusion of 'weightless substances' - heat and humidity.

From tables of Bessel functions ( [6] ) we find that for  $\varepsilon = 0.1$ ,  $x \leq 0.2$ . Passing to dimensional quantities (see [1] and [2]) we obtain an inequality for the time needed to set up a quasi-stationary state in the surface layer to an accuracy of up to 10%:  $t > t_0$  (where  $t_0 = 50, 500, 5000$  seconds, the height of the source being  $H \sim 10, 100, 1000$  m respectively, and where  $K_2/L$  1m/sec:  $K_2$  being the average magnitude of the coefficient of vertical turbulent diffusion outside the surface layer; and  $L$  is the height of the surface layer).

From this it follows that in many problems concerning the turbulent diffusion of a 'weightless' substance in the boundary layer of the atmosphere the process in the surface layer can in practice be thought of as quasi-stationary from the beginning. In this case we can suggest the following simple method for solving similar double-layer problems, taking an arbitrary law of change for the coefficient  $K(\zeta)$  in the surface layer.

In the layer  $0 \leq \zeta_0 < \zeta < 1$  the solution of the ordinary differential equation (1.1a) for  $v = 0$ , which satisfies the second condition of (1.3), is of the form

$$q_1^{(1)}(\tau, \zeta) = C_1 \left[ \frac{1}{\beta} + \int_{\zeta_0}^{\zeta} \frac{d\zeta}{K(\zeta)} \right]; \quad K(\zeta) \frac{\partial q_1^{(1)}}{\partial \zeta} = C_1, \quad (2.3)$$

where  $C_1$  is a parameter independent of  $\zeta$ . Eliminating  $C_1$  from these equations we obtain the relations

$$\left[ \frac{\partial q_1^{(1)}}{\partial \zeta} - \theta q_1^{(1)} \right]_{\zeta=1} = 0; \quad \frac{1}{\theta} = \frac{1}{\beta} + \int_{\zeta_0}^1 \frac{d\zeta}{K(\zeta)}. \quad (2.4)$$

on the boundary of the surface layer ( $\zeta = 1$ ,  $K(1) = 1$ ).

As a consequence of the equations  $\partial q_1^{(1)} / \partial \zeta|_{\zeta=1} = \partial q_1^{(2)} / \partial \zeta|_{\zeta=1}$ ;  $q_1^{(1)}(\tau, 1) = q_1^{(2)}(\tau, 1)$  (2.4) can be used as the boundary

condition at the height  $\zeta = 1$  to determine  $q_1^{(2)}(\tau, \zeta)$ , i.e. to solve equation (1.1a) in the half-plane  $\zeta > 1$  with the initial condition (1.2) and the first condition of (1.3). This problem is easily solved, since for  $\zeta > 1$  equation (1.1a) has constant coefficients. Having found  $q_1^{(2)}(\tau, \zeta)$  we find the  $C_1$  of (2.3) by using the equation  $C_1 = \partial q_1^{(2)} / \partial \zeta|_{\zeta=1}$ , and from (2.3) we can find the quasi-stationary solution  $q_1^{(v)}(\tau, \zeta)$  in the surface layer. We observe that the parameter of (2.4) is given by

$$1/\beta = 1/\beta = 1/D, \quad (2.5)$$

where  $D = \left[ \int_{\zeta_0}^1 K^{-1}(\zeta) d\zeta \right]^{-1}$  - is the experimentally determined total turbulent conductivity of the surface layer ([7]).

3. In [1] we described a method for reducing the problem (1.1) - (1.3) approximately to a boundary problem for an equation with constant coefficients (equation (1.1) with  $K(\zeta) \equiv 1$ ), by transferring the effect of a change in  $K(\zeta)$  in the layer  $\zeta_0 < \zeta < 1$  approximately into the boundary condition (1.3) at the height  $\zeta = \zeta_0$ . Condition (1.3) then takes the form

$$[\zeta \partial q / \partial \zeta + \nu q]_{\zeta=\zeta_0} = Bq(\tau, \zeta_0), \quad (3.1)$$

where

$$\frac{1}{B} = \begin{cases} 1/\nu + (1/\beta - 1/\nu)(\zeta_0^\nu - \nu \zeta_0)(1-\nu)^{-1} & \nu \neq 0; 1 \\ 1 + (1/\beta - 1)\zeta_0(1 - \ln \zeta_0) & \nu = 1 \\ 1/\beta - \ln \zeta_0 + \zeta_0 - 1 & \nu = 0^* \end{cases} \quad (3.2)$$

The initial condition (1.2) and the condition as  $\zeta \rightarrow +\infty$

\* In [1] the value of  $B$  is given only for  $\beta = \infty$ ,  $\nu > 0$ , however, the method described there can be used to find  $B$  for any  $\beta > 0$  and for the case  $\nu = 0$ .



are unchanged.

Let us find the connection between the solution  $q_2(\tau, \zeta)$  of this problem and the solution  $q_1^{(i)}(\tau, \zeta)$  ( $i = 1, 2$ ) of the problem with a quasi-stationary surface layer. In [1] we found the principal term of the asymptotic representation of  $q_2$ , analogous to (1.4):

$$q_2(\tau, \zeta) = \frac{e^{-(\chi_2 - \nu\tau)^2/4\tau}}{2\sqrt{\pi\tau}} \left[ e^{\lambda_1(\zeta - \zeta_0)} - e^{\lambda_2(\zeta - \zeta_0)} \frac{\lambda_1 + \nu - B}{\lambda_2 + \nu - B} \right] \left[ 1 + O\left(\frac{\chi_2^2}{\tau^3}\right) \right], \quad (3.3)$$

$$\chi_2 = \chi - \zeta_0.$$

Putting  $\nu = 0$ , let us form the expression  $\Lambda_0(\tau) = \partial q_2 / \partial \zeta|_{\zeta=1}$ , using the principal term of  $q_2$  given in (3.3). As  $\chi_2 / \tau \rightarrow 0$  we obtain in the limit

$$\lim \Lambda_0(\tau) = B / [1 + B(1 - \zeta_0)] = [1/\theta - \ln \zeta_0]^{-1} = \theta, \quad (3.4)$$

where  $\theta$  is defined by formula (2.4) for  $K(\zeta) = \zeta$ . This means that for large values of time  $\tau$  the solution  $q_2(\tau, \zeta)$  satisfies the condition (2.4) and, therefore, a quasi-stationary stage state is set up in the surface layer  $\zeta_0 < \zeta < 1$  for  $q_2$ .

We can verify at once that as  $\tau \rightarrow \infty$   $q_2(\tau, \zeta)$  and  $q_1^{(2)}(\tau, \zeta)$  tend to zero for  $1 < \zeta < \chi$  and  $\nu = 0$ , although from equations (3.2) and (2.4) with  $(\zeta) = \zeta$

$$\lim_{\tau \rightarrow \infty} \frac{q_2}{q_1^{(2)}} = \frac{1/B + \zeta - \zeta_0}{1/\theta + \zeta - 1} = 1, \quad (3.5)$$

Thus, for large values of  $\tau$ , and for  $\zeta > 1$ , the solution  $q_2$  and the solution  $q_1^{(2)}$  of the problem with a quasi-stationary surface layer are equivalent.

From this and the fact that, as we saw in section 1,  $q_1^{(2)}$  is equal to  $q(\tau, \zeta)$ , the solution of the 'exact' problem (1.1) - (1.3), when  $\nu = 0$  and as  $\tau \rightarrow \infty$ , it follows that

$q_2(\tau, \zeta)$  and  $q(\tau, \zeta)$  are equal for  $\zeta > 1$  and for large values of  $\tau$ .

In conclusion we note that when  $\nu > 0$  the principal terms of the solutions  $q_2(\tau, \zeta)$  and  $q_1^{(2)}(\tau, \zeta)$  defined by the expressions (3.3) and (1.4), (1.5) respectively, are no longer equal as  $\tau \rightarrow \infty$  with  $\zeta > 1$ . For, in the same way as for (3.5), it is not difficult to find that

$$\lim_{\tau \rightarrow \infty} \frac{q_2}{q_1^{(2)}} = \frac{\zeta - \zeta_0 + 2/(2B - \nu)}{\zeta - 1 + 2(\beta - \theta \zeta_0^\nu)/\nu(\beta + \theta \zeta_0^\nu)}.$$

This limit is not equal to unity for any  $\zeta \geq 1$  if  $B$  has the value given by equation (3.2). This corresponds to the deduction in section 1 that for  $\nu > 0$  and as  $\tau \rightarrow \infty$  the solution  $q(\tau, \zeta)$  of the 'exact' problem (1.1) - (1.3) does not become equal to the solution  $q_1^{(2)}(\tau, \zeta)$  for the quasi-stationary surface layer, and so the solution  $q_2(\tau, \zeta)$ , nearly equal to  $q(\tau, \zeta)$ , is different from  $q_1^{(2)}(\tau, \zeta)$  for  $\nu > 0$  and large values of  $\tau$ .

Thus, our simplifying assumption concerning a quasi-stationary state in the surface layer is legitimate only for the problem of the turbulent diffusion of a 'weightless' compound (aerosol). The solution of the problem of turbulent diffusion of a heavy aerosol ( $\nu > 0$ ) in the boundary layer of the atmosphere can be simplified by the method explained in [1] of substituting the coefficient of  $K(\zeta)$  in equation (1.1) by unity, and the boundary condition (1.3) by conditions (3.1) and (3.2) at the height  $\zeta = \zeta_0$ .

\* In [1] we established only that the flow  $K(\zeta) \partial q / \partial \zeta + \nu q$  of the function  $q(\tau, \zeta)$  was nearly equal to that of  $q_2(\tau, \zeta)$  at the height  $\zeta = \zeta_0$ .

### Bibliography

1. Karol' I.L. Ovliyanii prizemnogo sloya atmosfery na rasprostraneniye tyazheloy odnorodnoy primesi is vysotonogo tochechnogo istochnika. (On the effect of the surface layer of the atmosphere on the propagation of a heavy homogeneous compound from a high instantaneous point source) Publ. AS USSR, geophys. ser. No. 7, 1959.
2. Karol' I.L. O poluempiricheskoy teorii vertikal'noy turbulentnoy diffuzii v pogranichnom sloye atmosfery. (On the semi-empirical theory of vertical turbulent diffusion in the boundary layer of the atmosphere). Inzh- Fiz. Zhurnal, No. 4, 1960.
3. Berlyand, M.E. Raspredeleniye temperatury v atmosfery. (The distribution of temperature in the atmosphere). Meteorol. i gidrol., No. 6, 1949.
4. Berlyand, M.E. Izmenyeniye temperatury v prizemnom sloye atmosfery i predskazaniye zamorozkov. (Temperature changes in the surface layer of the atmosphere and forecasting frosts). Publ. AS USSR, geophys. ser., No. 2, 1953.
5. Berlyand, M.E. O transformatsii vozdukhnykh mass. (On the transformation of air masses). Tr. GGO, issue 41, 1953.
6. Watson, G.N. Teoriya besselevykh funktsiy, part 2 (The theory of Bessel functions). I.L., Moscow, 1949.
7. Gandin, L.S.; Leikhtman, D.L.; Matveyev, L.T.; Yudin, M.I. Osnovy dinamicheskoye meteorologii. (The essentials of dynamic Meteorology). Gidrometeoizdat, M. - L., 1955.

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